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On Multivariate Spline Systems

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There are two interesting tendencies in treating spline interpolation. The first is an *abstract spline theory within the framework of functional analysis*. In this connection the work of de Boor and Lynch [7], Jerome and Schumaker [18], Jerome and Varga [19], Atteia [2, 3], Anselone and Laurent [1], Sard [24], Aubin [4], Lucas [20], Scheffold [26] as well as Delvos and Schempp [9, 10] should be mentioned, the latter of which have introduced the concept of a *spline system*. The second tendency is concerned with *bi- and multivariate spline problems*. Here de Boor [6], Birkhoff, Schultz, and Varga [5], Ritter [22, 23], Schultz [27], Gordon [12], Mansfield [21], and Tippenhauer [29] should be mentioned. Some of these authors use tensor products for investigating multivariate problems.

It is the purpose of the present paper to generalize the concept of a *spline system* in order to consider multivariate spline interpolation problems by tensor product methods.

To achieve this, we first give in Section 1 the basic definition of a *spline system* (using weaker conditions than those of Delvos and Schempp [9]). Subsequently, *existence and uniqueness of a spline element* will be characterized separately, and two *approximation properties* of spline elements will be proved. The notion of a spline system given here enables us to give two *construction principles* for obtaining new spline systems from known ones: These methods are *additivity* and the use of *tensor products*. They are treated in Sections 2 and 3, respectively. These construction principles enable us to prove some results on tensor product spline systems as well as to deal with spline problems related to the work of Gordon [12] (see Section 4). Finally, in Section 5, we consider *interpolating spline systems* in one and two variables. Most of the spline interpolation problems that have been investigated can be described in the framework of interpolating spline systems.

Our notion of a spline system dispenses with a topology on the spline ground space. Thus we restrict our investigations on spline interpolation to those topics that may be dealt with within the framework of (algebraic) vector spaces as spline ground spaces. Topological aspects of spline systems and their tensor products will be treated elsewhere.

Some parts of the theory given in this paper have been developed in [16] under stronger conditions on the spline ground spaces.

1. Spline Systems and their Minimum Properties

A quadruple (X, P, U, H) will be called a *prespline system* provided that the following conditions hold:

(S1) X is a real (or complex) vector space, H is a real (resp. complex) prehilbert space with the scalar product $(h, \tilde{h}) \mapsto (h \mid \tilde{h})$ and the canonical norm

$$\| h \| = (h \mid h)^{1/2} \qquad (h \in H).$$

(S2) $P: X \to X$ is a linear idempotent mapping.
(S3) $U: X \to H$ is a linear mapping.

Let us denote prespline systems (and later on, spline systems) by script letters $\mathscr{P}, \mathscr{Q}, \ldots$, and let

$$\mathscr{I}_P(x) := \{t \in X: Pt = Px\} \qquad \text{for every} \quad x \in X.$$

In view of ordinary spline theory, X stands for the *spline ground space*, H replaces the *Hilbert space* L^2, P generalizes the *spline interpolation projection*, whereas U may be specialized, in a concrete case, to some *differential operator*.

Given a prespline system, we define the notions of a spline element and a spline system.

Definition 1. Let $\mathscr{P} := (X, P, U, H)$ be a prespline system and let $x \in X$. Then $s \in X$ is called a *spline element belonging to* x with respect to $\mathscr{P}$, if the following conditions hold:

(SE 0) $Ps = Px$ (i.e., $s \in \mathscr{I}_P(x)$).
(SE 1) $\| Us \| \leqslant \| Ut \|$ for all $t \in \mathscr{I}_P(x)$.

The *set of all spline elements belonging to* $x \in X$ with respect to $\mathscr{P}$ will be denoted by $\mathscr{S}_{\mathscr{P}}(x)$.

Definition 2. A prespline system $\mathscr{P} := (X, P, U, H)$ is called a *spline system*, if the following additional condition holds:

(S4) $\operatorname{Im} UP \perp \operatorname{Im} UP'$.

Here $\operatorname{Im} \Phi$ and below $\operatorname{Ker} \Phi$ designate the *image* and the *kernel*, respectively, of a linear mapping Φ, and P' is the *supplementary (projection) operator* of P having the following properties:

(P1) $P' = I - P$ (*I: identity on X*);
(P2) $P': X \to X$ is a linear idempotent mapping;
(P3) $\operatorname{Im} P' = \operatorname{Ker} P$, $\operatorname{Ker} P' = \operatorname{Im} P$.

We now characterize the validity of (S4).

THEOREM 1. *Let $\mathscr{P} = (X, P, U, H)$ be a prespline system. Then $\mathscr{P}$ satisfies* (S4) *if and only if for each $x \in X$ we have $Px \in \mathscr{S}_{\mathscr{P}}(x)$.*

Proof. Suppose that $Px \in \mathscr{S}_{\mathscr{P}}(x)$ for all $x \in X$. Assume there exists an $h_0 \in \operatorname{Im} UP$ and an $h_0' \in \operatorname{Im} UP'$ such that

$$(h_0 \mid h_0') = \alpha \neq 0.$$

Then $h_0' \neq 0$, and hence $(h_0' \mid h_0') = \beta > 0$. Since $h_0 \in \operatorname{Im} UP$ and $h_0' \in \operatorname{Im} UP'$, there exist $x_0 \in \operatorname{Im} P$ and $x_0' \in \operatorname{Im} P'$ which satisfy

$$Ux_0 = h_0 \qquad \text{and} \qquad Ux_0' = h_0', \quad \text{respectively.}$$

Consider $x_1 = x_0 - (\alpha/\beta) \cdot x_0' \in X$. By (P3) it follows that

$$Px_1 = Px_0 - \frac{\alpha}{\beta} Px_0' = Px_0 = x_0 .$$

By hypothesis, x_0 is a spline element belonging to x_1, i.e., $x_0 \in \mathscr{S}_{\mathscr{P}}(x_1)$. On the other hand

$$\begin{aligned}(Ux_1 \mid Ux_1) &= (Ux_0 \mid Ux_0) - \frac{\bar{\alpha}}{\beta} (Ux_0 \mid Ux_0') \\ &\quad - \frac{\alpha}{\beta} \overline{(Ux_0 \mid Ux_0')} + \frac{\alpha\bar{\alpha}}{\beta^2} (Ux_0' \mid Ux_0') \\ &= (Ux_0 \mid Ux_0) - \frac{|\alpha|^2}{\beta};\end{aligned}$$

hence

$$\| Ux_1 \| < \| Ux_0 \|.$$

This contradicts the fact that $x_0 = Px_1 \in \mathscr{S}_{\mathscr{P}}(x_1)$.

Conversely, let $\mathscr{P} = (X, P, U, H)$ be a spline system, and let $x \in X$. We have to show the relation $Px \in \mathscr{S}_{\mathscr{P}}(x)$ for all $x \in X$. (SE 0) is immediate. To prove (SE 1) we notice that for each $t \in \mathscr{I}_P(x)$ we have

$$t \in x + \operatorname{Ker} P = Px + \operatorname{Ker} P.$$

Thus one can express t in the form

$$t = Px + t' \qquad (t' \in \operatorname{Ker} P = \operatorname{Im} P').$$

From this we get

$$Ut = UPx + Ut' = UPx + UP't'.$$

(S4) yields

$$\| Ut \|^2 = \| UPx \|^2 + \| UP't' \|^2,$$

and hence

$$\| UPx \| \leqslant \| Ut \| \qquad \text{for any} \quad t \in \mathscr{I}_P(x).$$

This concludes the proof.

Now we investigate the *question of uniqueness of a spline element*. In general there is more than one spline element for a given $x \in X$. In the case of the *Lg-splines* this fact was pointed out by Jerome and Schumaker [18] and Jerome and Varga [19].

THEOREM 2. *Let $\mathscr{P} = (X, P, U, H)$ be a spline system. Then the following requirements are equivalent*:

(S5) Ker $U \cap$ Ker $P = (0)$.
(S5′) Ker $U \subset$ Im P.
(S5″) *For any $x \in X$, $\mathscr{S}_{\mathscr{P}}(x)$ consists of a single element, namely Px.*

In (S5), (0) designates the *trivial vector space* having only one element. If one of these conditions is satisfied, $\mathscr{P} = (X, P, U, H)$ will be said to be *unique*.

Proof. We show (S5) $\Rightarrow$ (S5′) $\Rightarrow$ (S5″) $\Rightarrow$ (S5).

Assume (S5) and let $x \in \operatorname{Ker} U$. By (P1) we have $x = Px + P'x$, hence, according to (S4),

$$0 = \| Ux \|^2 = \| UPx \|^2 + \| UP'x \|^2,$$

i.e., $UP'x = 0$. Thus $P'x \in \operatorname{Ker} U$. By (P3), $P'x \in \operatorname{Ker} P$, and (S5) yields $P'x = 0$. Therefore $x = Px$ and hence $x \in \operatorname{Im} P$. Thus Ker $U \subset$ Im P.

Now suppose Ker $U \subset$ Im P, and let $x \in X$. According to (SE 0) each spline element $s \in \mathscr{S}_{\mathscr{P}}(x)$ can be written in the form

$$s = Px + s' \qquad (s' \in \operatorname{Ker} P = \operatorname{Im} P').$$

Hence

$$\| Us \|^2 = \| UPx \|^2 + \| Us' \|^2.$$

Since $s \in \mathscr{S}_{\mathscr{P}}(x)$, it follows that $\| Us' \| = 0$; thus $s' \in \text{Ker } U$, i.e., $s' \in \text{Im } P$ by (S5′). On the other hand, $s' \in \text{Ker } P$. Thus we have $s' = 0$ so that (S5″) is satisfied.

Finally, suppose (S5″) holds, and let $x \in X$ be given. Assume there exists a $t \in \text{Ker } P \cap \text{Ker } U$, $t \neq 0$. Consider

$$s = Px + t \in X.$$

Obviously, $Ps = Px$. By (S4) we have

$$\| Us \|^2 = \| UPx \|^2 + \| Ut \|^2.$$

Since $\| Ut \|^2 = 0$, it follows that $s \in \mathscr{S}_{\mathscr{P}}(x)$, and, since $s \neq Px$, this contradicts (S5″).

COROLLARY 1. *Let $\mathscr{P} = (X, P, U, H)$ be a spline system and let $x \in X$. Then*:

(i) *$\mathscr{S}_{\mathscr{P}}(x)$ is a linear manifold in X*:

$$\mathscr{S}_{\mathscr{P}}(x) = Px + (\text{Ker } U \cap \text{Ker } P).$$

(ii) *The set $\mathscr{S}_{\mathscr{P}}$ of all spline elements (each of which belongs to an $x \in X$) is a subspace of X*:

$$\mathscr{S}_{\mathscr{P}} := \bigcup_{x \in X} \mathscr{S}_{\mathscr{P}}(x) = \text{Im } P \oplus (\text{Ker } U \cap \text{Ker } P).$$

(iii) *Given any $x \in X$, there exists a unique spline element in* Im P *belonging to x, namely Px, i.e.,*

$$\mathscr{S}_{\mathscr{P}}(x) \cap \text{Im } P = \{Px\}.$$

Px is called the standard spline element belonging to $x \in X$.

Remark. Given a spline system $\mathscr{P} = (X, P, U, H)$, then $\mathscr{P}' = (X, P', U, H)$ is a spline system, too. It is called the *supplementary spline system* with respect to $\mathscr{P}$. If Ker $U \neq (0)$, then at most one of these spline systems is unique.

A different notion of a *spline system*, which is based on the conditions (S1)–(S5) as well as on further topological properties concerning the spaces X and H and the mappings P and U was given by Delvos and Schempp [9]. Those *spline systems* may be considered as *unique topological spline systems* in our terminology. To construct new spline systems by tensor products (see Section 3) we need the notion of a spline system as given in definition 2. In general, tensor products of *spline systems* in the sense of Delvos and Schempp [9] fail to be *spline systems* in that sense (cf. Theorem 6). In the following, unless otherwise stated, we take as a "spline system" the concept defined in Definition 2.

Delvos and Schempp [9] proved two minimum properties which generalize known results (cf. de Boor and Lynch [7]). We shall prove *these minimum properties* under the weaker conditions demanded for our notion of a spline system.

THEOREM 3. *Let $\mathscr{P} = (X, P, U, H)$ be a (not necessarily unique) spline system; let $x \in X$; and let $s_0 \in \mathscr{S}_{\mathscr{P}}(x)$. Then the following minimum properties hold:*

(SE 1) $\| Us_0 \| \leqslant \| Ut \|$ *for all* $t \in \mathscr{I}_P(x)$.

(SE 2) $\| U(x - s_0)\| \leqslant \| U(x - s)\|$ *for all* $s \in \mathscr{S}_{\mathscr{P}}$.

Proof. (SE 1) is immediate. It turns out that (SE 2) may be considered as the *supplementary minimum property* with respect to (SE 1).

To show this, we first prove the following inequality for the standard spline element Px, which lies, by Theorem 1, in $\mathscr{S}_{\mathscr{P}}(x)$:

$$\| U(x - Px)\| \leqslant \| U(x - s)\| \qquad \text{for all} \quad s \in \operatorname{Im} P.$$

Indeed, given any $s_1 \in \operatorname{Im} P = \operatorname{Ker} P'$, there exists a $t_1 \in x + \operatorname{Ker} P'$ with

$$s_1 = x - t_1, \tag{1.1}$$

satisfying $P't_1 = P'x$. Since $\mathscr{P}' = (X, P', U, H)$ is a spline system, we have

$$\| UP'x \| \leqslant \| Ut_1 \|.$$

for t_1 according to (1.1). As $P' = I - P$, it follows that

$$\| U(x - Px)\| \leqslant \| U(x - s_1)\| \qquad \text{for all} \quad s_1 \in \operatorname{Im} P, \tag{1.2}$$

since s_1 was an arbitrary element of Im P.

Now each spline element $s_0 \in \mathscr{S}_{\mathscr{P}}(x)$ may be written in the form

$$s_0 = Px + t_0 \qquad (t_0 \in \operatorname{Ker} U \cap \operatorname{Ker} P).$$

From $Ut_0 = 0$ we get $Us_0 = UPx$, and thus

$$U(x - s_0) = U(x - Px). \tag{1.3}$$

By Corollary 1, (ii), any $s \in \mathscr{S}_{\mathscr{P}}$ has the unique representation

$$s = s_2 + t_2 \qquad (s_2 \in \operatorname{Im} P,\ t_2 \in \operatorname{Ker} U \cap \operatorname{Ker} P).$$

Since $Ut_2 = 0$, it follows that

$$U(x - s) = U(x - s_2) \qquad (s_2 \in \operatorname{Im} P).$$

From this, (1.2) and (1.3) we obtain

$$\| U(x - s_0)\| \leqslant \| U(x - s)\|$$

for any $s_0 \in \mathscr{S}_{\mathscr{P}}(x)$ and any $s \in \mathscr{S}_{\mathscr{P}}$.

2. Construction of Spline Systems by Additivity

In this section we are going to point out a construction principle for spline systems of the following kind: Given two spline systems $\mathscr{P}_1 = (X, P_1, U, H)$ and $\mathscr{P}_2 = (X, P_2, U, H)$, when is $(X, P_1 + P_2, U, H)$ also a spline system? The subsequent Theorem 4 will be applicable in connection with tensor products of spline systems (see Section 4).

Theorem 4. *Let $\mathscr{P}_1 = (X, P_1, U, H)$ and $\mathscr{P}_2 = (X, P_2, U, H)$ be two spline systems such that*

$$P_1 \cdot P_2 = P_2 \cdot P_1 = 0.$$

Then $\mathscr{R} := (X, P_1 + P_2, U, H)$, too, is a spline system. $\mathscr{R}$ is unique if at least one of the $\mathscr{P}_i$ is unique.

Proof. First we remark that $P_1 + P_2 \colon X \to X$ is an idempotent linear mapping. Since $P_1 \cdot P_2 = 0$, we have $\operatorname{Im} P_2 \subset \operatorname{Ker} P_1 = \operatorname{Im} P_1'$, and hence $\operatorname{Im} UP_2 \subset \operatorname{Im} UP_1'$. Since $\mathscr{P}_1$ is a spline system, by (S4)

$$\operatorname{Im} UP_2 \perp \operatorname{Im} UP_1. \tag{2.1}$$

Again by (S4), we get

$$(UP_1x \mid U(I - P_1)y) = 0 \qquad \text{for all} \quad (x, y) \in X \times X. \tag{2.2}$$

From (2.1) it follows that

$$(UP_1x \mid -UP_2y) = 0 \qquad \text{for all} \quad (x, y) \in X \times X. \tag{2.3}$$

Addition of (2.2) and (2.3) yields

$$(UP_1x \mid U(I - P_1 - P_2)y) = (UP_1x \mid U(P_1 + P_2)' y) = 0 \tag{2.4}$$

for all $(x, y) \in X \times X$. Similarly,

$$(UP_2x \mid U(P_1 + P_2)' y) = 0 \tag{2.5}$$

for all $(x, y) \in X \times X$. Hence

$$(U(P_1 + P_2)x \mid U(P_1 + P_2)' y) = 0$$

for all $(x, y) \in X \times X$, i.e.,

$$\operatorname{Im} U(P_1 + P_2) \perp \operatorname{Im} U(P_1 + P_2)'.$$

Thus $\mathscr{R}$ is a spline system.

To prove the *uniqueness part* of Theorem 4, we first show

$$\operatorname{Ker}(P_1 + P_2) \subset (\operatorname{Ker} P_1 \cap \operatorname{Ker} P_2). \tag{2.6}$$

Let $x \in \operatorname{Ker}(P_1 + P_2)$, i.e., $(P_1 + P_2)x = 0$. Then

$$0 = P_i(P_1 + P_2)x = (P_iP_1 + P_iP_2)x = P_i^2x = P_ix$$

$(i = 1, 2)$, and hence (2.6).

Suppose one of the spline systems $\mathscr{P}_1$ or $\mathscr{P}_2$ is unique, say $\mathscr{P}_1$. Since $\operatorname{Ker}(P_1 + P_2) \subset \operatorname{Ker} P_1$, it follows by (S5) that

$$\operatorname{Ker}(P_1 + P_2) \cap \operatorname{Ker} U = (0).$$

This concludes the proof.

3. Tensor Products of Spline Systems

In this section we use tensor products to get *multivariate spline systems.* We restrict ourselves to the bivariate case. By induction, our results may be extended to higher dimensions.

Our main result states that the tensor product of two spline systems is itself a spline system. Before proving it we briefly recall some facts concerning tensor products of vector spaces and linear mappings (cf., Greub [13]).

Let X and Y be real (or complex) vector spaces, and X_1, X_2 subspaces of X. Then:

(T1) $(X_1 \otimes Y) \cap (X_2 \otimes Y) = (X_1 \cap X_2) \otimes Y$.

(T1′) $(Y \otimes X_1) \cap (Y \otimes X_2) = Y \otimes (X_1 \cap X_2)$.

Given linear mappings

$$\varphi: X \to \tilde{X}, \qquad \psi: Y \to \tilde{Y},$$

where X, $\tilde{X}$, Y and $\tilde{Y}$ are real (or complex) vector spaces, following identities hold:

(T2) $\operatorname{Im}(\varphi \otimes \psi) = (\operatorname{Im} \varphi) \otimes (\operatorname{Im} \psi)$,

(T3) $\operatorname{Ker}(\varphi \otimes \psi) = (\operatorname{Ker} \varphi) \otimes Y + X \otimes (\operatorname{Ker} \psi)$.

Let $\hat{X}$ and $\hat{Y}$ be real (respectively complex) vector spaces, and let

$$\tilde{\varphi}: \tilde{X} \to \hat{X}, \qquad \tilde{\psi}: \tilde{Y} \to \hat{Y}$$

be linear mappings. Then we have

(T4) $(\tilde{\varphi} \otimes \tilde{\psi}) \cdot (\varphi \otimes \psi) = (\tilde{\varphi} \cdot \varphi) \otimes (\tilde{\psi} \cdot \psi)$.

We introduce now a *scalar product on the* (*algebraic*) *tensor product of two* (*real or complex*) *prehilbert spaces H and K* with scalar products $(h_1 \mid h_2)_H$ and $(k_1 \mid k_2)_K$, respectively. To this end, let $\{h_i\}_{i \in I}$ and $\{k_j\}_{j \in J}$ be, respectively, Hamel bases of H and K. Given two elements $z_1, z_2 \in H \otimes K$:

$$z_1 = \sum_{i \in I} \sum_{j \in J} a_{ij} \cdot x_i \otimes y_j\,,$$

$$z_2 = \sum_{k \in I} \sum_{l \in J} b_{kl} \cdot x_k \otimes y_l\,,$$

where each of the inequalities $a_{ij} \neq 0$, $b_{kl} \neq 0$ is satisfied for only a finite number of pairs $(i, j) \in I \times J$ resp. $(K, l) \in I \times J$, we define (cf. Dixmier [11], Schatten [25]):

$$(z_1 \mid z_2)_{H \otimes K} := \sum_{i \in I} \sum_{j \in J} \sum_{k \in I} \sum_{l \in J} a_{ij} \cdot \bar{b}_{kl} \cdot (x_i \mid x_k)_H (y_j \mid y_l)_K\,.$$

It turns out that $(z_1 \mid z_2)_{H \otimes K}$ is a scalar product on $H \otimes K$ which arises canonically from the scalar products given on H and K. The value of $(z_1 \mid z_2)_{H \otimes K}$ is independent of the choice of the Hamel bases $\{h_i\}_{i \in I}$ and $\{k_j\}_{j \in J}$. In writing $H \otimes K$, where H and K are prehilbert spaces, we mean the *algebraic tensor product of H and K provided with this canonical scalar product* which makes $H \otimes K$ a prehilbert space.

DEFINITION 3. Let $\mathscr{P} = (X, P, U, H)$ and $\mathscr{Q} = (Y, Q, V, K)$ be two spline systems (or prespline systems). Then

$$\mathscr{P} \otimes \mathscr{Q} := (X \otimes Y, P \otimes Q, U \otimes V, H \otimes K)$$

is called the *tensor product* of $\mathscr{P}$ and $\mathscr{Q}$.

The following lemma is obvious:

LEMMA 1. *Let $\mathscr{P} = (X, P, U, H)$ and $\mathscr{Q} = (Y, Q, V, K)$ be two prespline systems. Then their tensor product, $(X \otimes Y, P \otimes Q, U \otimes V, H \otimes K)$, is again a prespline system.*

THEOREM 5. *Given two spline systems $\mathscr{P} = (X, P, U, H)$ and $\mathscr{Q} = (Y, Q, V, K)$, then $\mathscr{P} \otimes \mathscr{Q} = (X \otimes Y, P \otimes Q, U \otimes V, H \otimes K)$ is also a spline system.*

Proof. By Lemma 1 we only have to verify the relation (S4) for the tensor product mappings:

$$\operatorname{Im}(U \otimes V) \cdot (P \otimes Q) \perp \operatorname{Im}(U \otimes V) \cdot (P \otimes Q)'.$$

To show this, we observe that, by (P3) and (T3),

$$\begin{aligned}\operatorname{Im}(P \otimes Q)' &= \operatorname{Ker} P \otimes Q = (\operatorname{Ker} P) \otimes Y + X \otimes (\operatorname{Ker} Q)\\ &= \operatorname{Ker} P \otimes \operatorname{Ker} Q \oplus \operatorname{Ker} P \otimes \operatorname{Ker} Q' \oplus \operatorname{Ker} P' \otimes \operatorname{Ker} Q,\end{aligned}$$

$\oplus$ denoting direct sum. Therefore, any $z \in \operatorname{Im}(P \otimes Q)'$ has a unique decomposition

$$z = \tilde{z}_1 + \tilde{z}_2 + \tilde{z}_3$$

such that

$$\tilde{z}_1 = \sum_{\mu=1}^{m} x_\mu \otimes y_\mu \qquad (x_\mu \in \operatorname{Ker} P, y_\mu \in \operatorname{Ker} Q),$$

$$\tilde{z}_2 = \sum_{\nu=1}^{n} \tilde{x}_\nu \otimes \tilde{y}_\nu' \qquad (\tilde{x}_\nu \in \operatorname{Ker} P, \tilde{y}_\nu' \in \operatorname{Ker} Q'),$$

$$\tilde{z}_3 = \sum_{\rho=1}^{r} \hat{x}_\rho' \otimes \hat{y}_\rho \qquad (\hat{x}_\rho' \in \operatorname{Ker} P', \hat{y}_\rho \in \operatorname{Ker} Q).$$

Let

$$\tilde{z}_0 = \sum_{\sigma=1}^{s} a_\sigma \otimes b_\sigma \qquad (a_\sigma \in \operatorname{Im} P, b_\sigma \in \operatorname{Im} Q)$$

be an arbitrary element of $\operatorname{Im} P \otimes Q = \operatorname{Im} P \otimes \operatorname{Im} Q$ (by (T2)).
Consider

$$\begin{aligned}((U \otimes V)\tilde{z}_0 \mid (U \otimes V)\tilde{z}_1)_{H \otimes K} &= \left(\sum_{\sigma=1}^{s} Ua_\sigma \otimes Vb_\sigma \,\middle|\, \sum_{\mu=1}^{m} Ux_\mu \otimes Vy_\mu\right)_{H \otimes K}\\ &= \sum_{\sigma=1}^{s} \sum_{\mu=1}^{m} (Ua_\sigma \mid Ux_\mu)_H \cdot (Vb_\sigma \mid Vy_\mu)_K .\end{aligned}$$

Since, by (S4), $Ua_\sigma \in \operatorname{Im} UP$ and $Ux_\mu \in \operatorname{Im} UP'$ for all $\sigma \in \{1,\dots,s\}$ and $\mu \in \{1,\dots,m\}$, as $\mathscr{P}$ is a spline system, we have

$$((U \otimes V)\,\tilde{z}_0 \mid (U \otimes V)\,\tilde{z}_1)_{H\otimes K} = 0$$

for all $\tilde{z}_0 \in \operatorname{Im} P \otimes Q$. In an analogous manner we can show that

$$((U \otimes V)\tilde{z}_0 \mid (U \otimes V)\tilde{z}_2)_{H\otimes K} = \sum_{\sigma=1}^{s}\sum_{\nu=1}^{n} (Ua_\sigma \mid U\tilde{x}_\nu)_H \cdot (Vb_\sigma \mid V\tilde{y}_\nu')_K = 0$$

for all $\tilde{z}_0 \in \operatorname{Im} P \otimes Q$, since $\mathscr{P}$ is a spline system. Similarly, by (S4) for $\mathscr{Q}$,

$$((U \otimes V)\tilde{z}_0 \mid (U \otimes V)\tilde{z}_3)_{H\otimes K} = \sum_{\sigma=1}^{s}\sum_{\rho=1}^{r} (Ua_\sigma \mid U\hat{x}_\rho')_H \cdot (Vb_\sigma \mid V\hat{y}_\rho)_K = 0$$

for all $\tilde{z}_0 \in \operatorname{Im} P \otimes Q$. Thus we have for all $z \in \operatorname{Im}(P \otimes Q)' = \operatorname{Ker} P \otimes Q$:

$$(U \otimes V)z \perp \operatorname{Im}(U \otimes V) \cdot (P \otimes Q),$$

i.e.,

$$\operatorname{Im}(U \otimes V) \cdot (P \otimes Q) \perp \operatorname{Im}(U \otimes V) \cdot (P \otimes Q)'.$$

The last theorem assures that given any $z = \sum_{\nu=1}^{n} x_\nu \otimes y_\nu \in X \otimes Y$, the element $(P \otimes Q)z$ lies in $\mathscr{S}_{\mathscr{P}\otimes\mathscr{Q}}(z)$. The standard spline element $(P \otimes Q)z$ can easily be computed if the *standard spline elements* Px_ν and Qy_ν $(1 \leqslant \nu \leqslant n)$ are known, since

$$(P \otimes Q)z = \sum_{\nu=1}^{n} Px_\nu \otimes Qy_\nu\,.$$

We now ask: when is $(P \otimes Q)z$ the only spline element belonging to z with respect to $\mathscr{P} \otimes \mathscr{Q}$?

THEOREM 6. *Let* $\mathscr{P} \otimes \mathscr{Q} = (X \otimes Y, P \otimes Q, U \otimes V, H \otimes K)$ *be the tensor product of two spline systems.* $\mathscr{P} \otimes \mathscr{Q}$ *is unique if and only if at least one of the following statements holds*:

(i) $X = (0)$ *or* $Y = (0)$.
(ii) $\operatorname{Ker} U = (0)$ *and* $\operatorname{Ker} V = (0)$.
(iii) $\operatorname{Ker} P = (0)$ *and* $\operatorname{Ker} Q = (0)$.
(iv) $\operatorname{Ker} U = (0)$, $\operatorname{Ker} P = (0)$ *and* $\operatorname{Ker} V \cap \operatorname{Ker} Q = (0)$.
(v) $\operatorname{Ker} V = (0)$, $\operatorname{Ker} Q = (0)$ *and* $\operatorname{Ker} U \cap \operatorname{Ker} P = (0)$.

Proof. We first show the sufficiency of any of the conditions (i)–(v). To do this we have to show that each of these conditions implies:

$$\operatorname{Ker}(U \otimes V) \cap \operatorname{Ker}(P \otimes Q) = (0). \tag{3.1}$$

By (T3),

$$\begin{aligned} \operatorname{Ker}(U \otimes V) &= (\operatorname{Ker} U) \otimes Y + X \otimes (\operatorname{Ker} V), \\ \operatorname{Ker}(P \otimes Q) &= (\operatorname{Ker} P) \otimes Y + X \otimes (\operatorname{Ker} Q). \end{aligned} \tag{3.2}$$

If $X = (0)$, then $\operatorname{Ker} U = (0)$, and hence $\operatorname{Ker}(U \otimes V) = (0)$; thus $\mathscr{P} \otimes \mathscr{Q}$ is unique. Similarly, if $Y = (0)$.

If $\operatorname{Ker} U = \operatorname{Ker} V = (0)$, then $\operatorname{Ker}(U \otimes V) = (0)$, and hence $\mathscr{P} \otimes \mathscr{Q}$ is unique.

If (iii) is satisfied, then $\operatorname{Ker}(P \otimes Q) = (0)$, and thus $\operatorname{Ker}(U \otimes V) \cap \operatorname{Ker}(P \otimes Q) = (0)$.

Suppose (iv) holds. Then:

$$\begin{aligned} \operatorname{Ker}(U \otimes V) &= X \otimes \operatorname{Ker} V, \\ \operatorname{Ker}(P \otimes Q) &= X \otimes \operatorname{Ker} Q, \end{aligned}$$

and by (T1′),

$$\operatorname{Ker}(U \otimes V) \cap \operatorname{Ker}(P \otimes Q) = X \otimes (\operatorname{Ker} V \cap \operatorname{Ker} Q) = (0).$$

If we assume (v), property (T1) yields

$$\operatorname{Ker}(U \otimes V) \cap \operatorname{Ker}(P \otimes Q) = (\operatorname{Ker} U \cap \operatorname{Ker} P) \otimes Y = (0).$$

The neccessity is proved as follows. Suppose (3.1) is satisfied. Assume none of the conditions

(A) $\operatorname{Ker} U = (0)$ and $\operatorname{Ker} P = (0)$,
(B) $\operatorname{Ker} U = (0)$ and $\operatorname{Ker} V = (0)$,
(C) $\operatorname{Ker} P = (0)$ and $\operatorname{Ker} Q = (0)$,
(D) $\operatorname{Ker} V = (0)$ and $\operatorname{Ker} Q = (0)$

holds. Then it follows that

$$(\operatorname{Ker} P) \otimes (\operatorname{Ker} V) + (\operatorname{Ker} U) \otimes (\operatorname{Ker} Q) \neq (0). \tag{3.3}$$

Since, by (3.2),

$$(\operatorname{Ker} P) \otimes (\operatorname{Ker} V) + (\operatorname{Ker} U) \otimes (\operatorname{Ker} Q) \subset \operatorname{Ker}(U \otimes V) \cap \operatorname{Ker}(P \otimes Q),$$

(3.3) contradicts (3.1). Hence at least one of the conditions (A), (B), (C) or (D) must be satisfied.

If (A) holds, then

$$\operatorname{Ker}(U \otimes V) \cap \operatorname{Ker}(P \otimes Q) = X \otimes (\operatorname{Ker} V \cap \operatorname{Ker} Q);$$

hence $X = (0)$ or $\operatorname{Ker} V \cap \operatorname{Ker} Q = (0)$. Thus (i) of (iv) is satisfied.

(B) is (ii) and (C) is (iii).

Finally, let (D) be satisfied. Then we have

$$\operatorname{Ker}(U \otimes V) \cap \operatorname{Ker}(P \otimes Q) = (\operatorname{Ker} U \cap \operatorname{Ker} P) \otimes Y.$$

This yields (i) or (v).

From Theorem 6 we see that the validity of the uniqueness condition (S5) for both (X, P, U, H) and (Y, Q, V, K) does not imply that their tensor product is also unique. This is why the tensor product of two spline systems in the sense of Delvos and Schempp [9] fails to be such a spline system.

4. Some Bivariate Spline Systems and their Minimum Properties

Using Theorems 4 and 5 we can construct some bivariate spline systems and prove corresponding minimum properties.

Corollary 2. *Let two spline systems* $\mathscr{P} = (X, P, U, H)$ *and* $\mathscr{Q} = (Y, Q, V, K)$ *be given. If* $z \in X \otimes Y$ *and* $s_0 \in \mathscr{S}_{\mathscr{P}\otimes\mathscr{Q}}(z)$, *then the following bivariate minimum properties hold*:

(i) $\|(U \otimes V) s_0\|_{H\otimes K} \leqslant \|(U \otimes V) s\|_{H\otimes K}$ *for every* $s \in \mathscr{I}_{P\otimes Q}(z)$,

(ii) $\|(U \otimes V)(z - s_0)\|_{H\otimes K} \leqslant \|(U \otimes V)(z - s)\|_{H\otimes K}$ *for every* $s \in \mathscr{S}_{\mathscr{P}\otimes\mathscr{Q}}$.

In particular, taking $z = \sum_{\tau=1}^{t} x_\tau \otimes y_\tau$, *we have the following properties of the bivariate standard spline element*:

(i′) $\|(U \otimes V)(\sum_{\tau=1}^{t} Px_\tau \otimes Qy_\tau)\|_{H\otimes K} \leqslant \|(U \otimes V)s\|_{H\otimes K}$ *for all* $s \in \mathscr{I}_{P\otimes Q}(z)$,

(ii′) $\|(U \otimes V)(\sum_{\tau=1}^{t} (x_\tau \otimes y_\tau - Px_\tau \otimes Qy_\tau))\|_{H\otimes K} \leqslant \|(U \otimes V)(z - s)\|_{H\otimes K}$ *for any* $s \in \mathscr{S}_{\mathscr{P}\otimes\mathscr{Q}}$.

Corollary 3. *Let* $\mathscr{P} = (X, P, U, H)$ *and* $\mathscr{Q} = (Y, Q, V, K)$ *be two spline systems. Then* $\mathscr{R} := (X \otimes Y, P \otimes Q + P' \otimes Q', U \otimes V, H \otimes K)$ *is also a spline system. Hence the following minimum properties hold, given any* $z \in X \otimes Y$ *and* $s_0 \in \mathscr{S}_{\mathscr{R}}(z)$:

(i) $\|(U \otimes V)s_0\|_{H\otimes K} \leqslant \|(U \otimes V)s\|_{H\otimes K}$ *for all* $s \in \mathscr{I}_{P\otimes Q+P'\otimes Q'}(z)$,

(ii) $\|(U \otimes V)(z - s_0)\|_{H\otimes K} \leqslant \|(U \otimes V)(z - s)\|_{H\otimes K}$ *for all* $s \in \mathscr{S}_{\mathscr{R}}$.

By a similar construction we get the following bivariate spline system, which is closely related to a problem considered by Gordon [12].

THEOREM 7. *Let $\mathscr{P} = (X, P, U, H)$ and $\mathscr{Q} = (Y, Q, V, K)$ be two spline systems, and $I\colon X \to X$ and $J\colon Y \to Y$ the identity mappings on X and Y, respectively. Then,*

$$\mathscr{T} := (X \otimes Y, I \otimes Q + P \otimes J - P \otimes Q, U \otimes V, H \otimes K)$$

is also a spline system. It is unique if both $\mathscr{P}$ and $\mathscr{Q}$ are unique. Let $z \in X \otimes Y$ and $s_0 \in \mathscr{S}_{\mathscr{T}}(z)$. Then

(i) $\|(U \otimes V)s_0\|_{H\otimes K} \leqslant \|(U \otimes V)s\|_{H\otimes K}$ *for all* $s \in \mathscr{I}_{I\otimes Q+P\otimes J-P\otimes Q}(z)$,

(ii) $\|(U \otimes V)(z - s_0)\|_{H\otimes K} \leqslant \|(U \otimes V)(z - s)\|_{H\otimes K}$ *for all* $s \in \mathscr{S}_{\mathscr{T}}$.

In particular, we have for the standard spline element,

(i′) $\|(U \otimes V)(I \otimes Q + P \otimes J - P \otimes Q)z\|_{H\otimes K} \leqslant \|(U \otimes V)s\|_{H\otimes K}$ *for all* $s \in \mathscr{I}_{I\otimes Q+P\otimes J-P\otimes Q}(z)$,

(ii′) $\|(U \otimes V)(P' \otimes Q')z\|_{H\otimes K} \leqslant \|(U \otimes V)(z - s)\|_{H\otimes K}$ *for all* $s \in \mathscr{S}_{\mathscr{T}}$.

Proof. We have only to show that the hypotheses of Theorem 4 are satisfied. To this end, observe that:

$$I \otimes Q + P \otimes J - P \otimes Q = I \otimes Q + P \otimes Q'.$$

Thus

$$(I \otimes Q) \cdot (P \otimes Q') = (I \cdot P) \otimes (Q \cdot Q') = 0$$

and

$$(P \otimes Q') \cdot (I \otimes Q) = (P \cdot I) \otimes (Q' \cdot Q) = 0.$$

By Theorem 5,

$$\mathscr{U} := (X \otimes Y, I \otimes Q, U \otimes V, H \otimes K)$$

and

$$\mathscr{V} := (X \otimes Y, P \otimes Q', U \otimes V, H \otimes K)$$

are spline systems, and hence $\mathscr{T}$ is a spline system.

Now we establish the uniqueness condition

$$\operatorname{Ker} U \otimes V \subset \operatorname{Im}(I \otimes Q + P \otimes J - P \otimes Q),$$

assuming that (X, P, U, H) and (Y, Q, V, K) satisfy (S5).

To prove this, let S and T be two linear projection operators mapping a vector space Z into itself. If $ST = 0$, then for any $z \in \operatorname{Im} T$:

$$z = Tz = Tz + Sz = (T + S)z,$$

i.e., $z \in \mathrm{Im}(T + S)$ and hence $\mathrm{Im}\, T \subset \mathrm{Im}(T + S)$. Similarly, if $TS = 0$, then $\mathrm{Im}\, S \subset \mathrm{Im}(T + S)$.

Applying this to our problem, we get

$$\mathrm{Im}\, P \otimes J \subset \mathrm{Im}(P \otimes J + I \otimes Q - P \otimes Q), \tag{4.1}$$
$$\mathrm{Im}\, I \otimes Q \subset \mathrm{Im}(P \otimes J + I \otimes Q - P \otimes Q). \tag{4.2}$$

Since $\mathscr{P}$ and $\mathscr{Q}$ are unique spline systems,

$$(\mathrm{Ker}\, U) \otimes Y \subset (\mathrm{Im}\, P) \otimes Y = \mathrm{Im}(P \otimes J),$$
$$X \otimes (\mathrm{Ker}\, V) \subset X \otimes (\mathrm{Im}\, Q) = \mathrm{Im}(I \otimes Q).$$

By (4.1) and (4.2),

$$\mathrm{Ker}(U \otimes V) = X \otimes (\mathrm{Ker}\, V) + (\mathrm{Ker}\, U) \otimes Y$$
$$\subset \mathrm{Im}(P \otimes J + I \otimes Q - P \otimes Q).$$

A connection with the work of Gordon [12] is obtained by specializing P and Q to be operators arising from *interpolation functionals*, and by providing X and Y with *appropriate topologies*. Topological results concerning spline systems will be given elsewhere, but a general treatment of spline systems arising from interpolation problems is given in the next section.

5. Interpolating Spline Systems

Most spline systems that occur in applications arise from certain *spline interpolation problems*; for example, from *natural polynomial spline functions* (cf. Greville [14]), L-splines (cf. Schultz and Varga [28], and Delvos and Schempp [10]) and *Lg-splines* (cf. Jerome and Schumaker [18]).

To deal with the corresponding concept of an interpolating spline system let us recall the following notion of an interpolation problem (cf. [17]).

Given $\mathscr{F} = (X, F; \Phi_1, ..., \Phi_m)$, where X is a real or complex vector space, F an m-dimensional subspace of X, and $\Phi_1, ..., \Phi_m$ linear functionals on X, one can raise the following *interpolation problem*:

Let $x \in X$. Does there exist an $f \in F$ satisfying

$$\Phi_\mu(f) = \Phi_\mu(x) \qquad (1 \leqslant \mu \leqslant m)? \tag{5.1}$$

Is it unique?

An answer is given by the following lemma (Davis [8]):

Lemma 2. *Let $\mathscr{F} = (x, F; \phi_1, ..., \phi_m)$ be an interpolation problem. A necessary and sufficient condition that for every $x \in X$ there exists exactly one $f \in F$ satisfying* (5.1) *is that the restrictions $\Phi_1 \mid F, ..., \Phi_m \mid F$ of the Φ_μ's to F are linearly independent in F^*, the algebraic dual of F.*

In this case $\mathscr{F}$ will be called a unique interpolation problem.

DEFINITION 4. $\mathscr{F} = (X, F; \Phi_1, ..., \Phi_m ; U, H)$ is called an *interpolating spline system*, provided:

(IS 1) $\mathscr{F}_0 := (X, F; \phi_1, ..., \phi_m)$ is a unique (real or complex) interpolation problem.

(IS 2) H is a real (respectively, complex) prehilbert space with scalar product $(h \mid \tilde{h})$ and canonical norm $\| h \| = (h \mid h)^{1/2}$.

(IS 3) $U: X \to H$ is a linear mapping.

(IS 4) The following orthogonality relation holds:

$$U(F) \perp U\left(\bigcap_{\mu=1}^{m} \operatorname{Ker} \Phi_\mu\right).$$

There is a close relation between interpolating spline systems and *finite spline systems* (i.e., spline systems (X, P, U, H) with $\dim \operatorname{Im} P < \infty$):

PROPOSITION 1. *Given an interpolating spline system $\mathscr{F} = (X, F; \Phi_1, ..., \Phi_m ; U, H)$, there is a uniquely determined finite spline system $\mathscr{P} = (X, P, U, H)$ generated by $\mathscr{F}$ in a canonical way. Conversely, any finite spline system $\mathscr{P} = (X, P, U, H)$ may be generated by some interpolating spline system which is, however, not uniquely determined.*

Proof. Let $\mathscr{F} = (X, F; \Phi_1, ..., \Phi_m ; U, H)$ be an interpolating spline system. Since the corresponding interpolation problem has a unique solution, there exists a unique dual base $\{f_1, ..., f_m\} \subset F$ satisfying

$$\Phi_\mu(f_\nu) = \delta_{\mu\nu} = \begin{Bmatrix} 0 & \text{for } \mu \neq \nu \\ 1 & \text{for } \mu = \nu \end{Bmatrix} \qquad \begin{pmatrix} 1 \leqslant \mu \leqslant m \\ 1 \leqslant \nu \leqslant n \end{pmatrix}.$$

Now

$$P: x \mapsto \sum_{\mu=1}^{m} f_\mu \cdot \Phi_\mu(x),$$

is a linear projection operator. It remains to show that (IS 4) implies (S4). Observe that $UF = \operatorname{Im} UP$. For the supplementary operator P' we have $\operatorname{Im} P' = \operatorname{Ker} P = \bigcap_{\mu=1}^{m} \operatorname{Ker} \Phi_\mu$. The last equality is proved as follows: Given $x \in \operatorname{Ker} P$, i.e.,

$$0 = Px = \sum_{\mu=1}^{m} f_\mu \cdot \Phi_\mu(x),$$

the linear independence of the f_μ's yields $\Phi_\mu(x) = 0$ $(1 \leqslant \mu \leqslant m)$; hence $x \in \bigcap_{\mu=1}^{m} \operatorname{Ker} \Phi_\mu$. Obviously $\bigcap_{\mu=1}^{m} \operatorname{Ker} \Phi_\mu \subset \operatorname{Ker} P$. Thus (IS 4) implies

$$\operatorname{Im} UP \perp \operatorname{Im} UP',$$

and so $\mathscr{P} = (X, P, U, H)$ is a spline system which is finite since $\dim \operatorname{Im} P = \dim F = m < \infty$.

Now let $\mathscr{P} = (X, P, U, H)$ be a finite spline system with $\dim \operatorname{Im} P = m < \infty$. Then there exist m linearly independent linear functionals $\varphi_1, ..., \varphi_m \in (\operatorname{Im} P)^*$. Let $\{f_1, ..., f_m\}$ be the corresponding dual base in Im P, and define $\Phi_\mu = \varphi_\mu \cdot P$. Then we have

$$P = \sum_{\mu=1}^{m} f_\mu \cdot \Phi_\mu ,$$

and, since

$$U(\operatorname{Im} P) = \operatorname{Im} UP \perp \operatorname{Im} UP' = U\left(\bigcap_{\mu=1}^{m} \operatorname{Ker} \Phi_\mu\right),$$

it follows that $\mathscr{F} = (X, \operatorname{Im} P; \Phi_1, ..., \Phi_m; U, H)$ is an interpolating spline system which generates $\mathscr{P}$.

Hence, given any interpolating spline system $\mathscr{F} = (X, F; \Phi_1, ..., \Phi_m; U, H)$, it has, by Lemma 2, an interpolation property corresponding to (5.1), and, in addition, two minimum properties hold, corresponding to (SE 1) and (SE 2).

Some bi- and multivariate spline interpolation problems can be studied in terms of interpolating spline systems, using tensor product methods. Existence and uniqueness for two-dimensional interpolation and the corresponding bivariate (and multivariate, respectively) minimum properties are consequences of:

THEOREM 8. *The tensor product of the interpolating spline systems* $\mathscr{F} = (X, F; \Phi_1, ..., \Phi_m; U, H)$ *and* $\mathscr{G} = (Y, G; \Psi_1, ..., \Psi_n; V, K)$, $\mathscr{F} \otimes \mathscr{G} := (X \otimes Y, F \otimes G; \Phi_\mu \otimes \Psi_\nu\colon 1 \leqslant \mu \leqslant m, 1 \leqslant \nu \leqslant n; U \otimes V, H \otimes K)$, *is also an interpolating spline system.*

Proof. As to (IS 1), by [15], the tensor product of two unique interpolation problems is a unique interpolation problem again. (IS 2) and (IS 3) are obvious. Therefore, one needs only to verify (IS 4), i.e.,

$$(U \otimes V)(F \otimes G) \perp (U \otimes V)\left(\bigcap_{\substack{1\leqslant\mu\leqslant m\\ 1\leqslant\nu\leqslant n}} \operatorname{Ker} \Phi_\mu \otimes \Psi_\nu\right).$$

To this end, consider the spline systems $\mathscr{P}_1 = (X, P_1, U, H)$ and $\mathscr{P}_2 = (Y, P_2, V, K)$ which are generated by the given interpolating spline systems $\mathscr{F}$ and $\mathscr{G}$, respectively. By (T2) we have $\operatorname{Im} P_1 \otimes P_2 = \operatorname{Im} P_1 \otimes \operatorname{Im} P_2$. By

$$\operatorname{Im}(P_1 \otimes P_2)' = \operatorname{Ker}(P_1 \otimes P_2) = \bigcap_{\substack{1\leqslant\mu\leqslant m\\ 1\leqslant\nu\leqslant n}} (\operatorname{Ker} \Phi_\mu \otimes \Psi_\nu),$$

and Theorem 5,

$$(U \otimes V)(F \otimes G) = \operatorname{Im}(U \otimes V) \cdot (P_1 \otimes P_2) \perp \operatorname{Im}(U \otimes V) \cdot (P_1 \otimes P_2)'$$
$$= (U \otimes V)\Big(\bigcap_{\substack{1 \leqslant \mu \leqslant m \\ 1 \leqslant \nu \leqslant n}} \operatorname{Ker} \Phi_\mu \otimes \Psi_\nu\Big).$$

This concludes the proof.

We show now how one can get *bivariate natural spline functions* using tensor product methods. For the one-dimensional case see Greville [14], de Boor and Lynch [7], and Delvos and Schempp [9].

Let k and m be positive integers, $k \leqslant m$, and let $-\infty < a < b < \infty$. Set

$$\mathbf{K}^{2,k}[a, b] := \{f : f \in C^{(k-1)}[a, b],\ f^{(k-1)} \text{ is absolutely continuous, } f^{(k)} \in \mathbf{L}^2[a, b]\}.$$

Given m real numbers x_μ satisfying

$$a \leqslant x_1 < x_2 < \cdots < x_m \leqslant b,$$

we define the linear interpolation functionals Φ_μ on $\mathbf{K}^{2,k}[a, b]$ as follows: $\Phi_\mu(f) = f(x_\mu)$, $\mu = 1, 2,..., m$. Furthermore, let $D^k f = f^{(k)}$ for $f \in \mathbf{K}^{2,k}[a, b]$. Denote by $S_m{}^k$ the *space of all natural polynomial spline functions of degree* $2k - 1$ associated with the nodes $\{x_1 ,..., x_m\}$ (cf. Greville [14], and Delvos and Schempp [9]). Then we have the following (one-dimensional) example of an interpolating spline system:

$$\mathscr{S} = (\mathbf{K}^{2,k}[a, b], S_m{}^k; \Phi_1 ,..., \Phi_m ; D^k, \mathbf{L}^2[a, b])$$

is a unique interpolating spline system.

This follows from [9].

Suppose now that two interpolating spline systems, corresponding to natural polynomial spline functions, are given:

$$\mathscr{S} = (\mathbf{K}^{2,k}[a, b], S_m{}^k; \Phi_1 ,..., \Phi_m ; D^k, \mathbf{L}^2[a, b])$$

as above, and

$$\mathscr{T} = (\mathbf{K}^{2,l}[c, d], T_n{}^l; \Psi_1 ,..., \Psi_n ; D^l, \mathbf{L}^2[c, d]),$$

where l and n are positive integers, $l \leqslant n$, $-\infty < c < d < \infty$, and the linear interpolation functionals $\Psi_1 ,..., \Psi_n$ are associated with nodes y_ν

satisfying $c \leqslant y_1 < y_2 < \cdots < y_n \leqslant d$. $T_n{}^l$ is the corresponding space of natural polynomial spline functions of degree $2l - 1$. Then Theorem 8 yields the following result:

$$\mathscr{S} \otimes \mathscr{T} := (\mathbf{K}^{2,k}[a, b] \otimes \mathbf{K}^{2,l}[c, d], S_m{}^k \otimes T_n{}^l; \Phi_\mu \otimes \Psi_\nu: 1 \leqslant \mu \leqslant m, \\ 1 \leqslant \nu \leqslant n; D^k \otimes D^l, \mathbf{L}^2[a, b] \otimes \mathbf{L}^2[c, d])$$

is an interpolating spline system.

Here we have the linear differential operator $D^k \otimes D^l = \partial^{k+1}/\partial x^k \, \partial y^l$.

We now summarize some properties of the interpolating spline system $\mathscr{S} \otimes \mathscr{T}$.

(i) *Interpolation property*:

Given the interpolating spline system $\mathscr{S} \otimes \mathscr{T}$ and a function $g \in \mathbf{K}^{2,k}[a, b] \otimes \mathbf{K}^{2,l}[c, d]$, there is one and only one $u \in S_m{}^k \otimes T_n{}^l$ satisfying

$$u(x_\mu, y_\nu) = g(x_\mu, y_\nu) \qquad (1 \leqslant \mu \leqslant m, 1 \leqslant \nu \leqslant n).$$

(ii) *Representation property*:

Let $\{s_1, ..., s_m\} \subset S_m{}^k$ and $\{t_1, ..., t_n\} \subset T_n{}^l$ be the dual bases corresponding to the functionals $\Phi_1 \mid S_m{}^k, ..., \Phi_m \mid S_m{}^k$ and $\Psi_1 \mid T_n{}^l, ..., \Psi_n \mid T_n{}^l$, respectively. Then the standard spline element $u \in S_m{}^k \otimes T_n{}^l$ belonging to $g \in \mathbf{K}^{2,k}[a, b] \otimes \mathbf{K}^{2,l}[c, d]$ has the following representation:

$$u = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} s_\mu \cdot t_\nu \cdot (\Phi_\mu \otimes \Psi_\nu)(g).$$

(iii) *Minimum properties*:

Let $g \in \mathbf{K}^{2,k}[a, b] \otimes \mathbf{K}^{2,l}[c, d]$. Then for any spline function s_0 belonging to g, we have:

(a) $$\int_a^b \int_c^d \left(\frac{\partial^{k+l}}{\partial x^k \, \partial y^l} s_0(x, y)\right)^2 dx\, dy \leqslant \int_a^b \int_c^d \left(\frac{\partial^{k+l}}{\partial x^k \, \partial y^l} s(x, y)\right)^2 dx\, dy$$

for all $s \in g + \left(\bigcap_{\substack{1 \leqslant \mu \leqslant m \\ 1 \leqslant \nu \leqslant n}} \operatorname{Ker} \Phi_\mu \otimes \Psi_\nu\right)$.

(b) $$\int_a^b \int_c^d \left(\frac{\partial^{k+l}}{\partial x^k \, \partial y^l} (g - s_0)(x, y)\right)^2 dx\, dy \\ \leqslant \int_a^b \int_c^d \left(\frac{\partial^{k+l}}{\partial x^k \, \partial y^l} (g - t)(x, y)\right)^2 dx\, dy$$

for all $t \in S_m{}^k \otimes T_n{}^l \oplus \left(\left(\bigcap_{\substack{1 \leqslant \mu \leqslant m \\ 1 \leqslant \nu \leqslant n}} \operatorname{Ker} \Phi_\mu \otimes \Psi_\nu\right) \cap \operatorname{Ker} D^k \otimes D^l\right)$.

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